Gravitational Lensing

An Introduction to the Main Concepts

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TOELT.AI

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Types of Lensing

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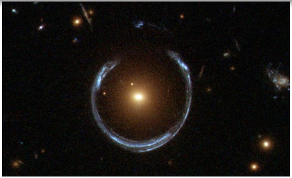

Credit: Wikipedia

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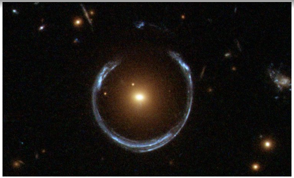

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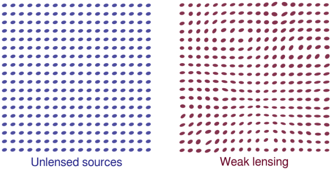

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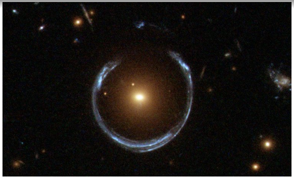
- **Weak Lensing:** Deflection impossible to detect for a single source. Effects are apparent by statistically averaging. Also for cosmology/dark matter.
- **Microlensing:** Lens object is usually smaller (like a star). Information obtained through brightness variations.



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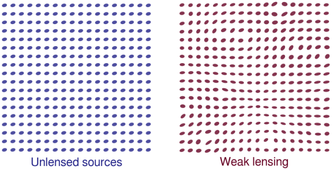

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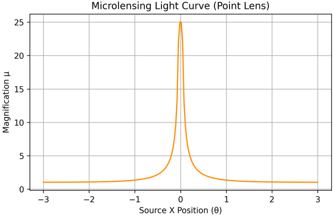
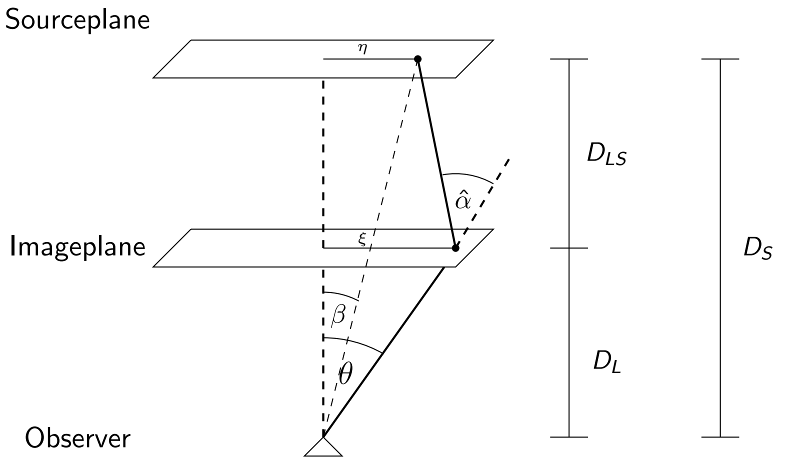


Gravitational Lensing Diagram



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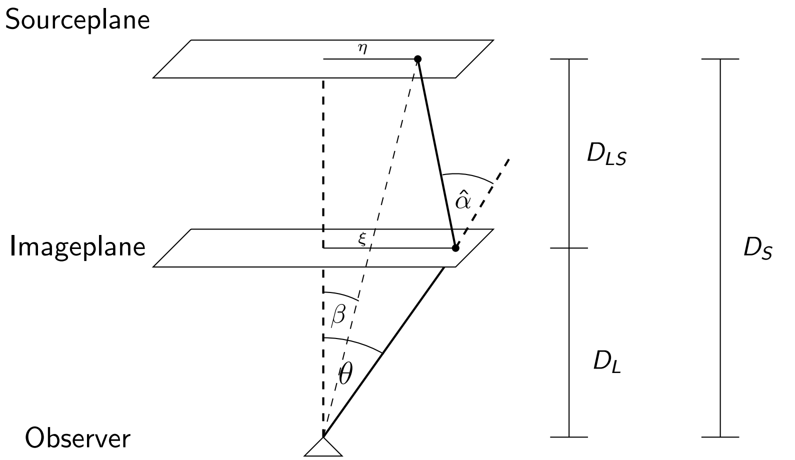
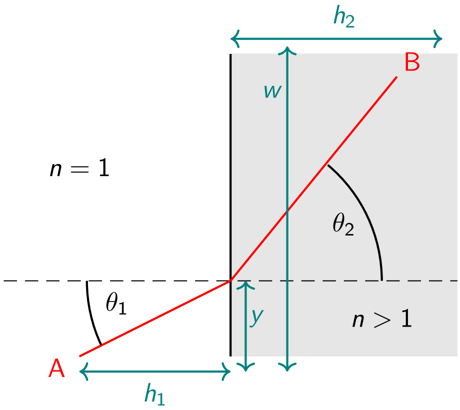


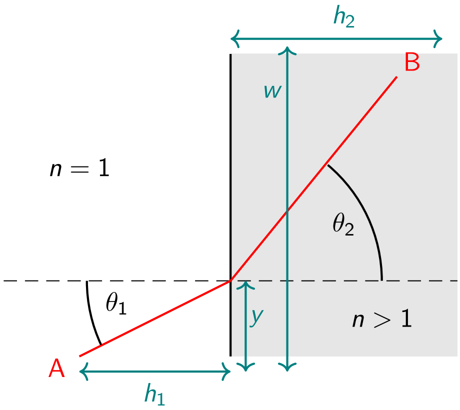
- Small Angles: $\vec{\theta} D_S = \vec{\beta} D_S + \vec{\hat{\alpha}} D_{LS}$ (with $\vec{\eta} = \vec{\beta} D_S$ and $\vec{\xi} = \vec{\theta} D_L$).
- **Lens Equation:** $\Rightarrow \boxed{\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta})}$, with $\vec{\alpha}(\vec{\theta}) = \vec{\hat{\alpha}}(\vec{\theta}) D_{LS} / D_S$.

Snell's Law



Snell's Law

- Total travel time:
  $t = \frac{1}{c}\sqrt{h_1^2 + y^2} + \frac{n}{c}\sqrt{h_2^2 + (w - y)^2}$



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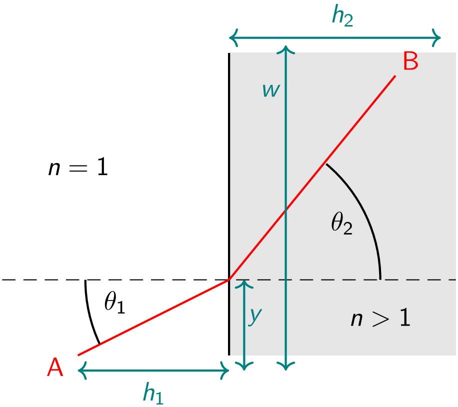
  $\frac{dt}{dy} = 0$
- Equivalence:
  $\frac{y}{\sqrt{h_1^2+y^2}} = n\frac{(w-y)}{\sqrt{h_2^2+(w-y)^2}}$



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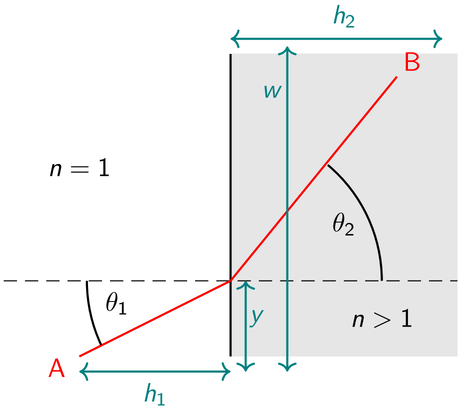
  $\frac{y}{\sqrt{h_1^2+y^2}} = n\frac{(w-y)}{\sqrt{h_2^2+(w-y)^2}}$
- $\Rightarrow \boxed{\sin\theta_1 = n\sin\theta_2}$



Light Rays in Space-Time

- Metric Tensor: $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

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- Weak Lens Perturbation:

$$\eta_{\mu\nu} \rightarrow g_{\mu\nu} = \text{diag}\left[1 + \frac{2\Phi}{c^2}, -(1 + \frac{2\Phi}{c^2}), -(1 + \frac{2\Phi}{c^2}), -(1 + \frac{2\Phi}{c^2})\right]$$

- Index of Refraction: $\Rightarrow n = \frac{c}{c'} \simeq 1 - \frac{2\,\phi}{c^2}$

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- We obtain a deflection angle $\Rightarrow \boxed{\hat{\vec{\alpha}}(b) = \frac{2}{c^2} \int_{-\infty}^{+\infty} \nabla_\perp \vec{\phi}\, dz}$

Light Rays in Space-Time

- Light ray deflection causes time delay $t_{\text{total}} = t_{\text{geom.}} + t_{\text{grav.}}$
- Time delay: $\tau(\vec{\theta}) = \frac{1+z_L}{c} \frac{D_L D_S}{D_{LS}} \left[\frac{1}{2}(\vec{\theta} - \vec{\beta})^2 - \Psi(\vec{\theta}) \right]$

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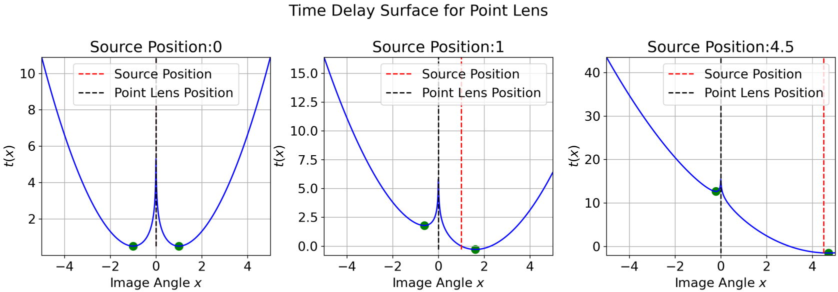
- Set extrema to find image locations:

$$\nabla_\theta \tau(\vec{\theta}) = \nabla_\theta \left[\frac{1}{2}(\vec{\theta}-\vec{\beta})^2 - \Psi(\vec{\theta})\right]$$

$$= (\vec{\theta}-\vec{\beta}) - \nabla_\theta \Psi(\vec{\theta}) = 0 \Rightarrow \text{Lens Eq.}$$



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- Measuring redshifts, angular positions to derive a lens mass model, assuming a cosmology and measuring the time delay allow us to estimate H_0.
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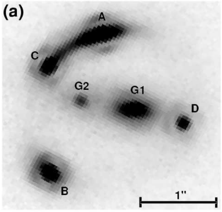
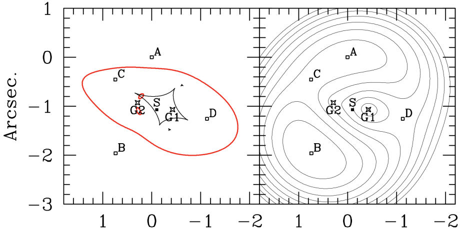


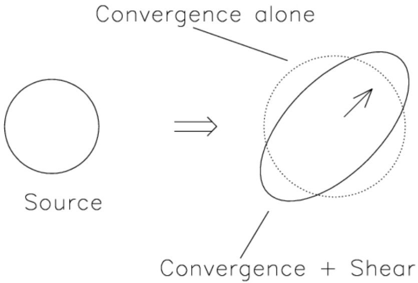
Lensing image (Koopmans et al. 2003).



Potential surface (Koopmans et al. 2003).

Convergence, Magnification and Shear

- The convergence and the shear are properties of the shape distortion which dictates the magnification.



Narayan & Bartelmann, 1995

Convergence:

- Define rescaled projection of potential onto lens plane:
  $\hat{\psi} = \frac{D_{LS}}{D_L D_S} \frac{2}{c^2} \int \phi(D_L \vec{\theta}, z) dz \quad \Rightarrow \quad \psi = \frac{D_L^2}{\xi_0^2} \hat{\psi}$
- Gradient gives: $\vec{\nabla}_x \psi(\vec{x}) = \vec{\alpha}(\vec{x})$
- Laplacian gives twice the convergence: $\nabla_x^2 \psi(\vec{x}) = 2\kappa(\vec{x})$

  where $\boxed{\kappa(\vec{x}) = \frac{\Sigma(\vec{x})}{\Sigma_{cr}}}$; $\Sigma_{cr} = \frac{c^2}{4\pi G} \frac{D_S}{D_L D_{LS}}$

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Shear:

- Distortion Matrix $A = \frac{d\vec{y}}{d\vec{x}} = \left(\delta_{ij} - \frac{\partial \alpha_i}{\partial \alpha_j}\right) = \left(\delta_{ij} - \frac{\partial^2 \psi}{\partial \psi_j^2}\right)$
- Split isotropic part off:
  $(A - \frac{1}{2} \text{tr} A \cdot I) = \begin{pmatrix} -(\psi_{11} - \psi_{22})/2 & -\psi_{12} \\ -\psi_{12} & (\psi_{11} - \psi_{22})/2 \end{pmatrix}$
- Shear $\gamma = (\gamma_1, \gamma_2) = ([\psi_{11} - \psi_{22}]/2, \ \psi_{12})$

Convergence, Magnification and Shear

- Jacobian written as $A = \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix}$
- Magnification: $\boxed{\mu = 1/\det A} = 1/\left[(1-\kappa^2) - \gamma^2\right]$

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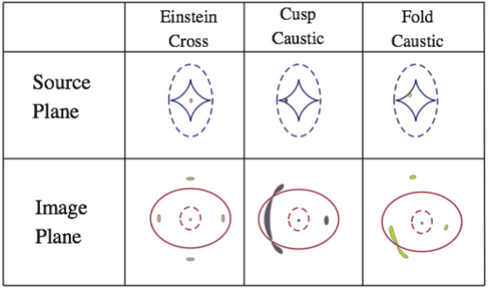

Credit: pbworks.com

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- Deflection angle vector: $\vec{\alpha}(b) = \frac{2GM}{c^2} \int_{\infty}^{\infty} \begin{pmatrix} x \\ y \end{pmatrix} \frac{dz}{(b^2+z^2)^{3/2}}$

$\Rightarrow \vec{\alpha}(b) = \frac{4GM}{c^2 b} \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}$

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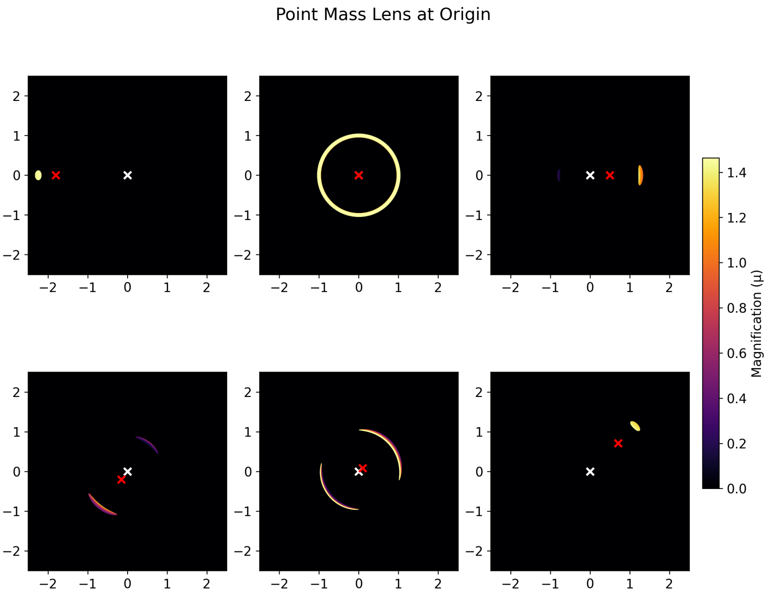
- Magnitude of deflection: $|\vec{\alpha}| = \frac{4GM}{c^2 b}$
- Twice the magnitude as predicted by Newton, since we account for the curvature of space-time. Time runs slower closer to a potential well, however the space around it is also curved.

Point Mass Lensing



Types of Lenses

Point Mass Lenses
$\hat{\alpha}(b) = \frac{4GM}{c^2 b}$

Axially Symmetric Lenses
$\hat{\alpha}(\xi) = \frac{4GM(\xi)}{c^2 \xi}$

Singular Isothermal Sphere (SIS)
Assume matter behaves like an ideal gas, thermal and hydrostatic equilibrium, in spherically symmetric potential.
$\rho(r) = \frac{\sigma_v^2}{2\pi G r^2}$
$\alpha(x) = x/|x|$

Softened Isothermal Sphere
Remove singularity with core radius
$\Psi = \sqrt{x^2 + x_c^2}$
$\alpha = x/\sqrt{x^2 + x_c^2}$

Navarro-Frenk-White profile (NFW)
Dark matter halo $\rightarrow$ reproduces calculations
$\rho(r) = \rho_s / \left[(r/r_s)\left(1 + r/r_s\right)^2\right]$

Extensions to background sources, elliptical lenses and more

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- Ellipse axes: $a = \frac{r}{1-\kappa-\gamma}$ and $b = \frac{r}{1-\kappa+\gamma}$

  $\Rightarrow$ Ellipticity $\epsilon = \frac{a-b}{a+b}$

  $\Rightarrow$ Split: $\epsilon_1 = \epsilon \cos 2\phi$ and $\epsilon_2 = \epsilon \sin 2\phi$

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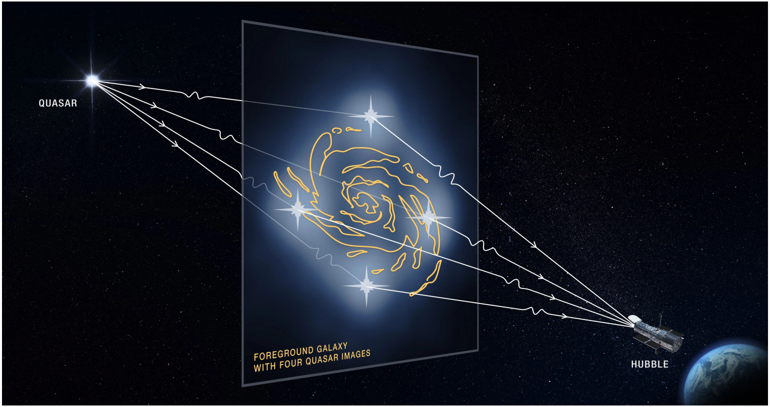
  $\Rightarrow$ Split: $\epsilon_1 = \epsilon \cos 2\phi$ and $\epsilon_2 = \epsilon \sin 2\phi$
- For source with intrinsic $\epsilon^{(s)}$, we get

  $$\epsilon_i = \epsilon_i^{(s)} + \gamma_i = \epsilon^{(s)} \begin{pmatrix} \cos 2\phi \\ \sin 2\phi \end{pmatrix} + \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}$$
- Averaging gives $\langle \epsilon \rangle = \langle \gamma \rangle$

Computational Methods and Techniques



Example of lensing galaxy. Credit: NASA

- **Main Lensing Aims:** Estimate the characteristics of the lensing model.
- **Process:** Specify coordinates then perform optimisation algorithms.

- **Parameterised Lens Modeling:** Lens equation $\vec{\beta}_i = \vec{\theta}_i - \vec{\alpha}(\vec{\theta}_i, \mathbf{p})$ for each image i.

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$$\chi^2_{\mathrm{src}} = \sum_i \left[(\vec{\beta} - \vec{\beta}_i)/\sigma_i\right]^2$$

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◇ *Full Imageplane:* Map to source plane with a full modeling of sources. Corresponding images are mapped to image plane where likelihood is computed.

- **Non-parametric Lens Modeling:**
  - Surface mass density as a sum over basis functions $\kappa(\vec{\theta'}) = \sum_j k_j f_j(\vec{\theta'})$ where θ' is the mass element position.
  - Lens equation $\vec{\beta} = \vec{\theta}_i - A_i \mathbf{k}$ for each image i.

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- Divide lens plane into grid with mass elements Σ_i and minimise statistic.
- Mass sheet degeneracy (adding a mass sheet and rescaling luminosities gives the same images)

  $\Rightarrow$ Add a regularisation term:

  $L(\mathbf{k}) = L_{\text{pos}}(\mathbf{k}) + \lambda_M L_{\text{reg,M}}(\mathbf{k})$

  with $L_{\text{pos}}(\mathbf{k}) = \sum_i \left[(\vec{\beta} - \vec{\beta}_i)/\sigma_i\right]^2$

  and $L_{\text{reg,M}}(\mathbf{k}) = \sum_{j,l}(\kappa_j - \kappa_l)$, where j, l are neighbours.

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- A typical likelihood is:

$$\mathcal{L}(\vec{k}, \vec{\beta}) \propto \exp\left(-\frac{1}{2}\sum_{i=1}^{N}\left(\vec{\theta}_i - A_i\vec{k} - \vec{\beta}\right)^{\top}\Sigma_i^{-1}\left(\vec{\theta}_i - A_i\vec{k} - \vec{\beta}\right)\right)$$

  which is a function of the residuals and Σ_i is the covariance matrix of the positional uncertainty.
- This form is the Mahalanobis distance.

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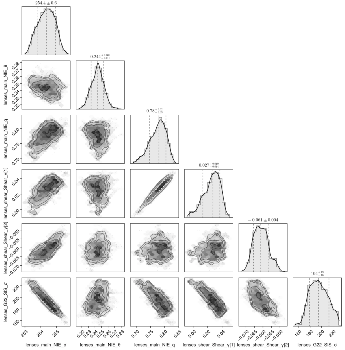
- Sample parameter space using Monte-Carlo chains, each time solving for the log-likelihood.
- A typical likelihood is:

$$\mathcal{L}(\vec{k},\vec{\beta}) \propto \exp\left(-\frac{1}{2}\sum_{i=1}^{N}\left(\vec{\theta}_i - A_i\vec{k} - \vec{\beta}\right)^{\top}\Sigma_i^{-1}\left(\vec{\theta}_i - A_i\vec{k} - \vec{\beta}\right)\right)$$

which is a function of the residuals and Σ_i is the covariance matrix of the positional uncertainty.

- This form is the Mahalanobis distance.
- Gives us $\Rightarrow$ uncertainties.



Credit: M. Lombardi, 2024

Finish

Thank you